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Discontinuous Behavior of Effective Transport Coefficients in Quasiperiodic Media

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We investigate the effective conductivity σ^* of a quasiperiodic medium in $\mathbb{R}^d$ and the discontinuous dependence, found in ref. 1, of σ^* on the wavelengths of the system. It was shown there, for example, that the effective conductivity $\sigma^*(k)$ for a layered medium with a one-dimensional local conductivity $\sigma_k(x) = A + \cos x + \cos kx$, $A > 2$, is discontinuous in k. An explicit class of higher-dimensional examples which exhibit the discontinuity is constructed here. The conductivity $\sigma^*(k, L)$ of a sample of length L in one dimension as $L \to \infty$ is also analyzed and shown to have a plateau structure for any irrational k well approximated by rationals.

KEY WORDS: Quasiperiodic media; effective conductivity; discontinuous dependence on wavelengths; sample-size dependence.

1. INTRODUCTION

Recently we observed[(1)] that classical transport coefficients of a quasiperiodic medium in $\mathbb{R}^d$ with a conductivity $\sigma(\mathbf{x})$ and/or potential $V(\mathbf{x})$ depend discontinuously on the frequencies of the quasiperiodicity. For example, when $\sigma_k(x) = A + \cos x + \cos kx$, $A > 2$, in $\mathbb{R}^1$, the effective conductivity

$$\sigma^*(k) = \lim_{L \to \infty} \sigma^*(k, L), \qquad [\sigma^*(k, L)]^{-1} = \frac{1}{2L} \int_{-L}^{L} [1/\sigma_k(x)]\, dx$$

has the same value $\bar{\sigma}$ for all *irrational* k, but depends on k for k *rational.* In fact, $\sigma^*(k)$ is discontinuous at rational k and is continuous at irrational

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k. The discontinuity arises in the infinite-volume limit in the computation of $\sigma^*(k)$; $\sigma^*(k, L)$ is, of course, continuous in k for finite L.

To see the origin of the discontinuity in one dimension, we note that a quasiperiodic function containing two frequencies can be written as $\sigma_{\mathsf{k}}(x) = \hat{\sigma}(k_1 x, k_2 x) = \hat{\sigma}(\mathsf{k}x)$, where $\mathsf{k} = (k_1, k_2)^T$ is a two-by-one matrix and $\hat{\sigma}(x, y)$ is periodic in both x and y of period 1. (We write k as a column vector here to be consistent with later notation.) For the example above, $\hat{\sigma}(x, y) = A + \cos x + \cos y$. The integral for $\sigma^*(k)$ is over a trajectory of the flow $(\dot{\omega}_1, \dot{\omega}_2) = (k_1, k_2)$, $(\omega_1, \omega_2) \in T^2$, the 2-torus, which is ergodic only when $k = k_2/k_1$ is irrational. In this case, the limiting integration is over all of T^2 with respect to Lebesgue measure. In the rational case, however, the trajectory degenerates to a closed orbit on T^2, over which the integral is in general different from its value over all of T^2. Similarly, a quasiperiodic function with n frequencies, $n \geqslant 2$, can be written as $\sigma_{\mathsf{k}}(x) = \hat{\sigma}(\mathsf{k}x)$, where $\mathsf{k} = (k_1, \ldots, k_n)^T$ and $\hat{\sigma}$ is of period 1 in each variable, and the corresponding flow will be on the n-torus T^n.

There is no such general argument for $d \geqslant 2$, where there is no explicit formula for the effective conductivity tensor $\boldsymbol{\sigma}^*$. We therefore construct here a class of two-component media which exhibit the discontinuity. In these systems the local conductivity $\sigma_{\mathsf{k}}(\mathbf{x})$, taking values σ_1 and σ_2, is the restriction of some periodic function on $\mathbb{R}^n$, $n \geqslant d+1$, to a d-dimensional subspace whose basis vectors form the n by d matrix k. In particular, for $d = 2$ we take a plane slice of a three-dimensional checkerboard of cubes with conductivities σ_1 and σ_2. When the corresponding 2-parameter "flow" on T^3 is ergodic (k *"irrational"*), $\boldsymbol{\sigma}^*$ of the resulting quasiperiodic medium is invariant under interchange of the components σ_1 and σ_2. The Keller interchange equality[2-6] then yields the surprising result that $\det(\boldsymbol{\sigma}^*)$ has the same value $\sigma_1 \sigma_2$ for *all* irrational planes. To obtain a discontinuity, we consider a particular rational angle for which we show that $\det(\boldsymbol{\sigma}^*)$ has a value different than $\sigma_1 \sigma_2$. Generalizations of the checkerboard for which $\det(\boldsymbol{\sigma}^*)$ is the same for irrational k are also discussed.

For $d \geqslant 3$ the interchange equality becomes an inequality.[4,6] We still obtain the discontinuity in $\det(\boldsymbol{\sigma}^*)$ by bounding its value for a particular rational k away from its possible values for irrational k.

To obtain greater physical understanding of the discontinuity, observe that if in our one-dimensional example we introduce a "phase" $\boldsymbol{\omega} = (\omega_1, \omega_2)$ by setting $\sigma_k(x, \boldsymbol{\omega}) = A + \cos(x + \omega_1) + \cos(kx + \omega_2)$, then $\sigma^*(k, \boldsymbol{\omega})$ will depend on $\boldsymbol{\omega}$ for k rational but not for k irrational. For the $d = 2$ checkerboard example one can see this as well by observing that for k irrational the relative volume fractions p_1 and $p_2 = 1 - p_2$ of σ_1 and σ_2 are independent of phase, with $p_1 = p_2 = \frac{1}{2}$, while for k rational they depend on phase. In other words, the discontinuity in $\boldsymbol{\sigma}^*$ arises from a discontinuity in the microgeometry, as characterized by the volume fractions. It is surprising that even after averaging, say in the one-dimensional example, $\sigma^*(k, \boldsymbol{\omega})$, over $\boldsymbol{\omega}$ with respect to Lebesgue measure on T^2, the result $\sigma^*_{\mathrm{av}}(k)$ for rational k is *still* unequal to $\bar{\sigma}$. In fact, we prove, for general $\hat{\sigma}$ on T^2 with effective conductivity $\bar{\sigma}$ for irrational k, that $\sigma^*_{\mathrm{av}}(k) \geqslant \bar{\sigma}$, for all rational k. In higher dimensions, we prove a natural generalization of this inequality, namely, that $\sigma^*_{\mathrm{av}}(\mathsf{k})$ is upper semicontinuous in k.

Since the discontinuity arises only in the infinite-volume limit, it is important to ask what might be observed in an experiment where one must work with a finite sample of size L. We investigate this question, when the variation in σ is one dimensional, for irrational k that are very well approximated by rationals k_n, with $k_n \to k$ as $n \to \infty$. In this case $\sigma^*(k, L)$ has "plateaus" with values $\sigma^*(k_n)$ over appropriate ranges of L. The smaller $|k - k_n|$ is, the longer the corresponding plateau, which we interpret in terms of the continued fraction expansion of k.

In ref. 7 we analyze the plateaus and their consequences in any dimension using more general arguments which apply as well to diffusion in $\mathbb{R}^d$ obeying $d\mathbf{X}_t = -\nabla V(\mathbf{X}_t)\,dt + d\mathbf{W}_t$, where $\mathbf{W}_t$ is standard Brownian motion, $\mathbf{X}_0 = 0$, and V is quasiperiodic with frequency matrix k. In this case, the effective diffusion tensor

$$\mathsf{D}^*(\mathsf{k}) = \lim_{t \to \infty} \mathbf{D}^*(\mathsf{k}, t), \qquad D^*_{ij}(\mathsf{k}, t) = E[X^i_t X^j_t]/t$$

exhibits the discontinuity like $\sigma^*(\mathsf{k})$, and $\mathbf{D}^*(\mathsf{k}, t)$ has plateaus in t like $\sigma^*(\mathsf{k}, L)$.

It is interesting to compare our classical transport with that of quantum transport in quasiperiodic potentials. This is a field with much current activity.[8-11] In particular, it has been shown that the nature of the wave functions satisfying the time-dependent Schrödinger equation with potential $q(x) = \cos x + \alpha \cos(kx + \theta)$ depends very sensitively on the rationality of k. The interpretation in that case is in terms of interference, leading in some cases to localization—something that does not occur classically. Nevertheless, we see here that classical transport, too, depends sensitively on the commensurability of the frequencies characterizing the system.

2. FORMULATION

Let $\hat{\sigma}(\boldsymbol{\omega})$ be a function on the unit n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, $\boldsymbol{\omega} \in T^n$, which we identify with the obvious periodic function on $\mathbb{R}^n$. We will similarly use "$\wedge$" to indicate other functions on T^n. We define the local conductivity field $\sigma_{\mathsf{k}}(\mathbf{x}, \boldsymbol{\omega})$, $x \in \mathbb{R}^d$, via

$$\sigma_{\mathsf{k}}(\mathbf{x}, \boldsymbol{\omega}) \equiv \hat{\sigma}(\boldsymbol{\omega} + \mathsf{k}\mathbf{x}) \tag{2.1}$$

where k is an n by d matrix, $\mathsf{k} = [\mathbf{k}_1^T, \dots, \mathbf{k}_d^T]$, $\mathbf{k}_i \cdot \mathbf{k}_j = 0$, $i \neq j$, $\mathbf{k}_i \in \mathbb{R}^n$, and

$$\mathsf{k}\mathbf{x} = \sum_{\alpha=1}^{d} \mathbf{k}_\alpha x_\alpha \tag{2.2}$$

Given $\sigma_{\mathsf{k}}(\mathbf{x}, \boldsymbol{\omega})$, we consider the electric field $\mathbf{E}_j(\mathbf{x}, \boldsymbol{\omega}) = \hat{\mathbf{E}}_j(\boldsymbol{\omega} + \mathsf{k}\mathbf{x})$ and current field $\mathbf{J}_j(\mathbf{x}, \boldsymbol{\omega}) = \mathbf{J}_j(\boldsymbol{\omega} + \mathsf{k}\mathbf{x})$ satisfying

$$\mathbf{J}_j(\mathbf{x}, \boldsymbol{\omega}) = \sigma_{\mathsf{k}}(\mathbf{x}, \boldsymbol{\omega})\, \mathbf{E}_j(\mathbf{x}, \boldsymbol{\omega}) \tag{2.3}$$

$$\nabla \cdot \mathbf{J}_j = 0 \tag{2.4}$$

$$\nabla \times \mathbf{E}_j = 0 \tag{2.5}$$

$$⨍_{\mathbb{R}^d} d\mathbf{x}\, \mathbf{E}_j(\mathbf{x}, \boldsymbol{\omega}) = \mathbf{e}_j \tag{2.6}$$

where $\mathbf{e}_j$ is a unit vector in the jth direction in $\mathbb{R}^d$, and the integral in (2.6) is an infinite-volume average of $\mathbf{E}_j(\mathbf{x}, \boldsymbol{\omega})$ over $\mathbb{R}^d$.

We shall be most interested in two-component media, arising from

$$\hat{\sigma}(\boldsymbol{\omega}) = \sigma_1 \hat{\chi}_1(\boldsymbol{\omega}) + \sigma_2 \hat{\chi}_2(\boldsymbol{\omega}) \tag{2.7}$$

where σ_1, $\sigma_2 > 0$, and the indicator functions $\hat{\chi}_i(\boldsymbol{\omega})$, $i = 1, 2$, satisfy $\hat{\chi}_1 + \hat{\chi}_2 = 1$. Due to the absence of smoothness in this case, Eqs. (2.4) and (2.5) should be understood to hold weakly in an appropriate subspace of $L^2(T^n, d\boldsymbol{\omega})$,[12,13] where $\partial/\partial x_i$ is identified with the generator of translations in the direction of $\mathbf{k}_i$.

The effective conductivity tensor $\sigma^* \equiv \sigma^*(\mathsf{k}) \equiv \sigma^*(\mathsf{k}, \boldsymbol{\omega})$ is defined via

$$\sigma^* \mathbf{e}_j = ⨍_{\mathbb{R}^d} d\mathbf{x}\, \sigma_{\mathsf{k}}(\mathbf{x}, \boldsymbol{\omega})\, \mathbf{E}_j(\mathbf{x}, \boldsymbol{\omega}) \tag{2.8}$$

It is symmetric. We remark that σ^* can also be defined in terms of the diffusion process in a random medium with generator $\frac{1}{2}\nabla \cdot (\sigma(\mathbf{x}, \boldsymbol{\omega})\nabla)$. In one dimension, if $\boldsymbol{\omega} \to \boldsymbol{\omega} + \mathsf{k}x$, $x \in \mathbb{R}$, is ergodic with respect to $d\boldsymbol{\omega}$ on T^n,

$$[\sigma^*]^{-1} = \int_{T^n} d\boldsymbol{\omega}\, [\hat{\sigma}(\boldsymbol{\omega})]^{-1} \tag{2.9}$$

while for finite lengths,

$$[\sigma^*(L, \boldsymbol{\omega})]^{-1} = \frac{1}{2L} \int_{-L}^{L} [\sigma_{\mathsf{k}}(x, \boldsymbol{\omega})]^{-1}\, dx \tag{2.10}$$

The convergence of (2.10) to (2.9) as $L \to \infty$ is in $L^2(T^n, d\boldsymbol{\omega})$. Note hat (2.10) provides a suitable definition of the finite-length conductivity for any k, necessary irrational. [For the definition of $\sigma^*(L, \boldsymbol{\omega})$ in any dimension, see ref. 13.]

The "flow" $\boldsymbol{\omega} \to \boldsymbol{\omega} + \mathsf{k}\mathbf{x} = \tau_{\mathbf{x}}^{(\mathsf{k})}\boldsymbol{\omega}$, $\mathbf{x} \in \mathbb{R}^d$, on T^n leaves invariant Lebesgue measure $d\boldsymbol{\omega}$ on T^n. It is also ergodic relative to $d\boldsymbol{\omega}$ when the equations $\mathbf{k}_1 \cdot \mathbf{j} = 0, \ldots, \mathbf{k}_d \cdot \mathbf{j} = 0$ have no simultaneous integral solutions $\mathbf{j} \in \mathbb{Z}^n$, $\mathbf{j} \neq 0$.[14] We say that k is "irrational" in this case, i.e., when $\tau_{\mathbf{x}}^{(\mathsf{k})}$ is ergodic, and is "rational" otherwise. In particular, when k is irrational $\sigma^*(\mathsf{k})$ is almost surely constant as a function of $\boldsymbol{\omega}$. When $n = 2$, $d = 1$, and $\mathsf{k} = \mathbf{k} = [k_1, k_2]^T$, k is "irrational" when k_2/k_1 is irrational. When $n > d + 1$, k can have various degrees of rationality, depending on the dimension of the ergodic components of $\tau_{\mathbf{x}}^{(\mathsf{k})}$. In general, σ^* will depend upon $\boldsymbol{\omega}$ only through the "ergodic component" to which $\boldsymbol{\omega}$ belongs.

3. THE CHECKERBOARD AND ITS GENERALIZATIONS

We now construct explicit examples of systems for which $\sigma^*(\mathsf{k})$ is discontinuous in k. First we look at the one-dimensional case $\sigma_k(x) = \hat{\sigma}(x, kx)$, where $\hat{\sigma}$ is a checkerboard on T^2. Then we consider its higher-dimensional analogs and a generalization of these models which yields a class of media which exhibit the discontinuity in the same way as the checkerboards.

3.1. $d = 1$

Let $\hat{\sigma}(\boldsymbol{\omega})$ on the unit 2-torus T^2 be defined as follows. Divide T^2 into four equal squares with the common vertex (1/2, 1/2). On the squares let $\hat{\sigma}(\boldsymbol{\omega})$ take the positive values σ_1 or σ_2 in a checkerboard arrangement, with, say, σ_2 on the square nearest the origin. Extend this by periodicity to the whole plane $\mathbb{R}^2$, and define

$$\sigma_k(x) = \sigma_k(x, \boldsymbol{\omega} = \mathbf{0}) = \hat{\sigma}(x, kx) \tag{3.1}$$

which we visualize as the restriction of $\hat{\sigma}$ to a trajectory of slope k passing through the origin; see Fig. 1.

Now for $\sigma_k(x)$ in (3.1),

$$[\sigma^*(k)]^{-1} = p_1(k)/\sigma_1 + p_2(k)/\sigma_2 \tag{3.2}$$

where $p_j(k)$ is the proportion of length that the line of slope k in $\mathbb{R}^2$ spends in regions (squares) where $\hat{\sigma} = \sigma_j$, $j = 1, 2$, for the above described checkerboard. For further simplicitly we assume that $\sigma_1 = 1$ and $\sigma_2 = \infty$.

Then we have the following result.

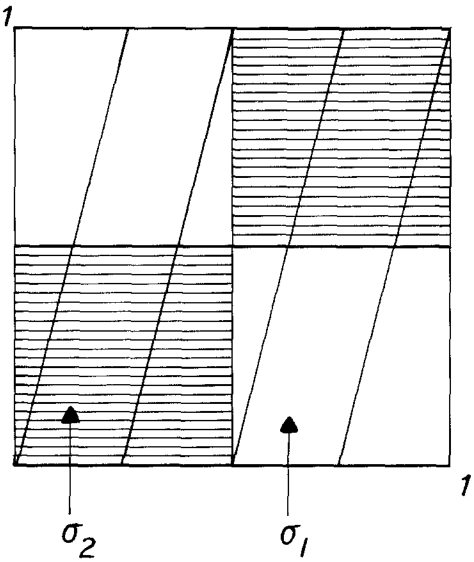

Fig. 1. One-dimensional medium defined by the restriction of the checkerboard $\hat{\sigma}$ of σ_1 and σ_2 on T^2 to the trajectory of slope $k = 4$.

Theorem 3.1. For $\sigma_k(x) = \hat{\sigma}(x, kx)$ with $\hat{\sigma}$ the above checkerboard of squares $\sigma_1 = 1$ and $\sigma_2 = \infty$, and $k > 0$,

$$1/\sigma^*(k) = \begin{cases} 1/2, & k \text{ irrational} \\ 1/2 - 1/(2pq), & k = p/q,\ p \text{ and } q \text{ odd, relatively prime integers} \\ 1/2, & k = p/q,\ \text{otherwise} \end{cases} \tag{3.3}$$

The proof is provided by D. Barsky in the Appendix to this paper.

3.2. $d = 2$

The analog of the checkerboard for T^3 is obtained by dividing it into eight equal cubes with common vertex (1/2, 1/2, 1/2), with $\hat{\sigma}$ taking the values σ_1 and σ_2 in a checkerboard fashion. Given k and this $\hat{\sigma}$, (2.1) defines $\sigma_{\mathrm{k}}(\mathbf{x}, \boldsymbol{\omega})$, which is quasiperiodic when k is irrational and periodic when the coordinates of both $\mathbf{k}_1 = (k_{11}, k_{21})$ and $\mathbf{k}_2 = (k_{12}, k_{22})$ are rational.

As indicated in the Introduction, we obtain a discontinuity in $\det(\sigma^*)$ by first examining it for k irrational, and then by exhibiting particular rationals for which its values are separated from those in the irrational case.

Our principal tool will be the Keller interchange equality[2,6]: let $\sigma^*(\sigma_1, \sigma_2)$ be the effective conductivity tensor of any ergodic two-component material and let $\sigma^*(\sigma_2, \sigma_1)$ be the effective conductivity tensor of the material with σ_1 and σ_2 interchanged. Then

$$\sigma_1^*(\sigma_1, \sigma_2)\,\sigma_2^*(\sigma_2, \sigma_1) = \sigma_1\sigma_2 \tag{3.4}$$

where $\sigma_1^* \leqslant \sigma_2^*$ are the eigenvalues of the symmetric matrix σ^*. The following observation allows (3.4) to provide information about $\det(\sigma^*)$.

Lemma 3.1. For k irrational, the quasiperiodic medium $\sigma_{\mathrm{k}}(\mathbf{x}, \boldsymbol{\omega})$ arising from the checkerboard on T^3 satisfies

$$\sigma^*(\mathrm{k}; \sigma_1, \sigma_2) = \sigma^*(\mathrm{k}; \sigma_2, \sigma_1) \tag{3.5}$$

i.e., $\sigma^*(\mathrm{k})$ is invariant under the interchange of the components.

Proof. Suppose k is irrational, then $\sigma^*(\mathrm{k})$ is independent of $\boldsymbol{\omega}$ almost surely. However, interchange of the components σ_1 and σ_2 is induced by $(\omega_1, \omega_2, \omega_3) \mapsto (\omega_1 + \frac{1}{2}, \omega_2, \omega_3)$ on T^3. Thus, $\sigma^*(k)$ is interchange invariant.

As an immediate consequence of (3.4) and Lemma 3.1, we have the following:

Theorem 3.2. Let $\sigma_{\mathrm{k}}(\mathbf{x}, \boldsymbol{\omega}) = \hat{\sigma}(\boldsymbol{\omega} + \mathrm{k}\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, where $\hat{\sigma}$ is a checkerboard of σ_1 and σ_2 on T^3. Then for all irrational k,

$$\det(\sigma^*(\mathrm{k})) = \sigma_1\sigma_2 \tag{3.6}$$

We now obtain the discontinuity. Let the cube nearest the origin in T^3 $(\simeq [0, 1)^3)$ have conductivity σ_2. Consider the plane passing through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and then translate it downward so that it passes through $(0, 0, 3/4)$. Let k_0 span this plane and let $\boldsymbol{\omega}_0 = (0, 0, 3/4)$. The resulting pattern $\sigma_{\mathrm{k}_0}(\mathbf{x}, \boldsymbol{\omega}_0)$ is a periodic array of six-pointed stars with the central hexagon of σ_1 (see Fig. 2), which is clearly "isotropic," $\sigma_{ij}^*(\mathrm{k}_0, \boldsymbol{\omega}_0) = \sigma^*(\mathrm{k}_0, \boldsymbol{\omega}_0)\,\delta_{ij}$, due to the sixfold symmetry about the center of the hexagon. However, this array is *not* interchange invariant, since $p_1 = 3/4$, while $p_2 = 1/4$, which indicates that we should *not* expect that $\det(\sigma^*(\mathrm{k}_0, \boldsymbol{\omega}_0)) = \sigma_1\sigma_2$.

Lemma 3.2. There exist σ_1 and σ_2 such that for the resulting $\hat{\sigma}$ and k_0, $\boldsymbol{\omega}_0$ as above

$$\det(\boldsymbol{\sigma}^*(\mathrm{k}_0, \boldsymbol{\omega}_0)) \neq \sigma_1\sigma_2 \tag{3.7}$$

Proof. Since $\sigma_{\mathrm{k}_0}(\mathbf{x}, \boldsymbol{\omega}_0)$ is isotropic

$$\det(\boldsymbol{\sigma}^*(\mathrm{k}_0, \boldsymbol{\omega}_0)) = (\sigma^*(\mathrm{k}_0, \boldsymbol{\omega}_0))^2 \tag{3.8}$$

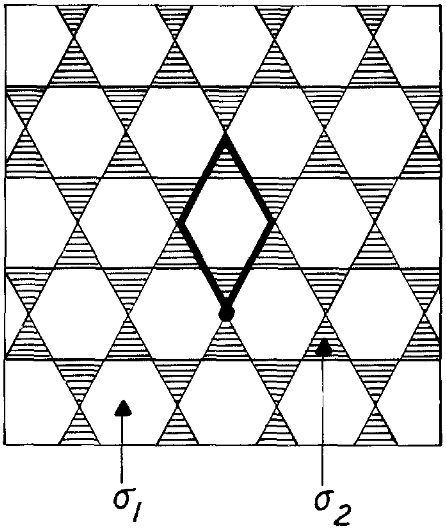

Fig. 2. Two-dimensional medium defined by the restriction of the checkerboard $\hat{\sigma}$ of σ_1 and σ_2 on T^3 to the plane defined by k_0 and ω_0. A period cell has been outlined, and the darkened point at its bottom corresponds to (0, 0, 3/4) in T^3.

By the well-known arithmetic mean upper bound[13]

$$\sigma^*(\mathrm{k}_0, \omega_0) \leqslant p_1\sigma_1 + p_2\sigma_2 = \tfrac{3}{4}\sigma_1 + \tfrac{1}{4}\sigma_2 \tag{3.9}$$

But

$$(\tfrac{3}{4}\sigma_1 + \tfrac{1}{4}\sigma_2)^2 < \sigma_1\sigma_2 \tag{3.10}$$

when

$$\sigma_1 < \sigma_2 < 9\sigma_1 \tag{3.11}$$

Thus

$$\det(\sigma^*(\mathrm{k}_0, \omega_0)) < \sigma_1\sigma_2 \tag{3.12}$$

when (3.11) is satisfied. ▮

Theorem 3.2 and Lemma 3.2 together yield a discontinuity in $\det(\sigma^*(k))$ at $\mathrm{k} = \mathrm{k}_0$. Since $\det(\sigma^*)$ is a continuous function of σ^*, we have the following:

Corollary 3.1. Let σ_1 and σ_2 be as in Lemma 3.2. Then $\sigma^*(\mathrm{k})$ is discontinuous at $\mathrm{k} = \mathrm{k}_0$.

We have constructed here only one example of a rational k for which the discontinuity can be proven. When the denominators in the rational numbers in k are much larger, so that p_1 and p_2 are both very close 1/2, the simple proof given above will not work, as much tighter bounds on σ^* would be required. Nevertheless, we expect that $\sigma^*(\mathsf{k})$ is discontinuous at "most" rational k.

We remark that not all periodic media arising from the checkerboard are isotropic like the "stars." Consider the plane that contains the x axis and (0, 1, 1). The resulting pattern is infinite strips of width 1/2 alternating in σ_1 and σ_2. The principal directions of this medium are parallel and perpendicular to the strips. Parallel to the strips, the corresponding eigenvalue of σ^* is $\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2$, and perpendicular to the strips, it is $[1/(2\sigma_1) + 1/(2\sigma_2)]^{-1}$.

In one dimension, the value of $\sigma^*(\mathsf{k})$ is independent of k when k is irrational, for general $\hat{\sigma}$ on T^n, $n \geqslant 2$. For $d \geqslant 2$, $\sigma^*(\mathsf{k})$ for general $\hat{\sigma}$ on T^n may depend on k for k irrational (as well as rational). The following example illustrates this for $d = 2$. Let $\hat{\sigma}$ on T^3 be a two-component medium composed of a thin cylindrical tube of σ_1 in the ω_3-direction in the center of T^3, surrounded by σ_2. Further assume $\sigma_1 \gg \sigma_2$. Now let $\mathsf{k}_\perp$ span an irrational plane which is almost perpendicular to the cylinder axis. The resulting medium is a quasiperiodic array of disks of σ_1 embedded in σ_2 which are only very slightly elongated in one direction, so that $\sigma^*(\mathsf{k}_\perp)$ is presumably very close to being isotropic, i.e., a multiple of the identity. However, for $\mathsf{k}_{\|}$ spanning an irrational plane which is almost parallel to the cylinder axis, the resulting medium consists of a quasiperiodic array of very long parallel spikes of σ_1 embedded in σ_2 (in the same volume fraction as the disks in the $\mathsf{k}_\perp$ case). In this case $\sigma^*(\mathsf{k}_{\|})$ is presumably highly anisotropic, with the degree of anisotropy increasing as either σ_1 is increased or the $\mathsf{k}_{\|}$ plane is further aligned with the ω_3 axis.

3.3. $d \geqslant 3$

For $d \geqslant 3$, the inequality

$$\sigma_i^*(\sigma_1, \sigma_2)\, \sigma_j^*(\sigma_2, \sigma_1) \geqslant \sigma_1 \sigma_2 \tag{3.13}$$

replaces (3.4), for all pairs of eigenvalues σ_i^* and σ_j^*. Schulgasser[4] first proved (3.13), and Kohler and Papanicolaou[6] proved a more general form of it. Since Lemma 3.1 holds for T^n as well as T^3, slight manipulation of (3.13) yields the following result.

Theorem 3.3. Let $\sigma_{\mathsf{k}}(\mathbf{x}, \boldsymbol{\omega}) = \hat{\sigma}(\boldsymbol{\omega} + \mathsf{k}\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, $d \geqslant 3$, where $\hat{\sigma}$ is a checkerboard of σ_1 and σ_2 on T^n, $n \geqslant d + 1$. Then for all irrational k,

$$\det(\sigma^*(\mathsf{k})) \geqslant (\sigma_1 \sigma_2)^{d/2} \tag{3.14}$$

To establish the discontinuity for $d \geqslant 3$, we again use the bound

$$\sigma_i^*(\mathrm{k}_0) \leqslant p_1\sigma_1 + p_2\sigma_2 \tag{3.15}$$

for any k_0 and i. Inequality (3.15) yields

$$\det(\sigma^*(\mathrm{k}_0)) < (\sigma_1\sigma_2)^{d/2} \tag{3.16}$$

at least when

$$\sigma_1 p_1 = \sigma_2 p_2 \tag{3.17}$$

and

$$p_1 p_2 < 1/4 \tag{3.18}$$

At these k_0 we have, in view of (3.16) and Theorem 3.3, a discontinuity in $\det(\sigma^*(\mathrm{k}))$ if σ_1 and σ_2 are chosen so that (3.17) holds.

We remark that whenever interchange of σ_1 and σ_2 in the ambient environment $\hat{\sigma}$ on $\mathbb{R}^n$ is induced by a change in realization $\boldsymbol{\omega} \to \boldsymbol{\omega}'$, which, in fact, can be assumed to be a translation, the conclusions of Theorem 3.2 for $d=2$ or Theorem 3.3 for $d \geqslant 3$ hold.

4. PHASE AVERAGING

We first consider phase averaging in one dimension for a medium $\sigma_{\mathrm{k}}(x, \boldsymbol{\omega}) = \hat{\sigma}(\boldsymbol{\omega} + \mathrm{k}\mathbf{x})$, $\boldsymbol{\omega} \in T^n$. Define

$$\sigma_{\mathrm{av}}^*(\mathrm{k}) = \int_{T^n} \sigma^*(\mathrm{k}, \boldsymbol{\omega})\, d\boldsymbol{\omega} \tag{4.1}$$

where $\sigma^*(\mathrm{k}, \boldsymbol{\omega})$ is the effective conductivity of $\sigma_{\mathrm{k}}(\mathbf{x}, \boldsymbol{\omega})$. Also let $[\bar{\sigma}]^{-1}$ be given by the right side of (2.9). Then we have the following:

Theorem 4.1. For $d=1$,

$$\sigma_{\mathrm{av}}^*(\mathrm{k}) \geqslant \bar{\sigma} \tag{4.2}$$

Furthermore, equality holds in (4.2) if and only if $\sigma^*(\mathrm{k}, \boldsymbol{\omega})$ is independent of $\boldsymbol{\omega}$ (almost everywhere with respect to Lebesgue measure on T^n).

Proof. Suppose k is rational. Let $\langle \cdot \rangle_{\boldsymbol{\omega}}$ denote normalized averaging over the trajectory $\tau_x^{(\mathrm{k})}\boldsymbol{\omega}$, $x \in \mathbb{R}$, on T^n. Then

$$\sigma_{\mathrm{av}}^*(\mathbf{k}) = \int_{T^n} \frac{1}{u(\boldsymbol{\omega})}\, d\boldsymbol{\omega} \tag{4.3}$$

where $u(\boldsymbol{\omega}) = \langle 1/\hat{\sigma} \rangle_{\boldsymbol{\omega}}$. By Jensen's inequality,

$$\sigma_{\mathrm{av}}^*(\mathbf{k}) \geqslant 1 \Big/ \int_{T^n} u(\boldsymbol{\omega})\, d\boldsymbol{\omega} = \bar{\sigma} \tag{4.4}$$

where equality holds in (4.4) if and only if $u(\boldsymbol{\omega})$ is independent of $\boldsymbol{\omega}$ (almost everywhere in Lebesgue measure on T^n).

The statement below (4.2) shows that, typically, phase averaging preserves the discontinuity of $\sigma^*(\mathsf{k})$ at rational k in one dimension. While we have not *proven* that the discontinuity is generally present in higher dimensions for $\boldsymbol{\sigma}_{\mathrm{av}}^*(\mathsf{k})$, which is the analog of (4.1) for $d \geqslant 1$, we still have the following result.

Theorem 4.2. $\sigma_{\mathrm{av}}^*(\mathsf{k})$ is upper semicontinuous in k.

Proof. We use an alternative variational formula for σ^* [6]: For any $\mathbf{e} \in \mathbb{R}^d$

$$\mathbf{e} \cdot \sigma_{\mathrm{av}}^*(\mathsf{k}) \cdot \mathbf{e} = \inf_{\mathbf{E} \in \mathscr{E}} \int_{T^n} d\boldsymbol{\omega}\, \hat{\sigma}(\boldsymbol{\omega})\, \hat{\mathbf{E}}^2(\boldsymbol{\omega}) \tag{4.5}$$

where $\mathscr{E}$ is the set of fields satisfying (2.5) and (2.6) with $\mathbf{e}_j$ replaced by $\mathbf{e}$. In terms of potential fields f on T^n, (4.5) can be written as

$$\mathbf{e} \cdot \sigma_{\mathrm{av}}^*(\mathsf{k})\, \mathbf{e} = \inf_{f \in H^1} \int_{T^n} d\boldsymbol{\omega}\, \sigma(\boldsymbol{\omega})(1 + D_{\mathbf{e}}^{\mathsf{k}} f)^2 \tag{4.6}$$

where

$$H^1 = \{ f \in L^2(T^n, d\boldsymbol{\omega}) \mid D_{\mathbf{e}}^{\mathsf{k}} f \in L^2(T^n, d\boldsymbol{\omega}), \forall \mathbf{e} \in \mathbb{R}^d \}$$

and $D_{\mathbf{e}}^{\mathsf{k}}$ is the generator of the translation subgroup $\tau_{t\mathbf{e}}^{(\mathsf{k})}$, $t \in \mathbb{R}$. We have thus characterized $\mathbf{e} \cdot \sigma_{\mathrm{av}}^*(\mathsf{k}) \cdot \mathbf{e}$ as an infimum of continuous functions of k, so that it is upper semicontinuous in k.

5. BEHAVIOR OF $\sigma^*[k, L]$ AS $L \to \infty$: PLATEAUS

In this section we examine the effective conductivity of a sample extending from $x = 0$ to $x = L$,

$$[\sigma^*(k, L)]^{-1} = \frac{1}{L} \int_0^L [\sigma_k(x)]^{-1}\, dx \tag{5.1}$$

where $\sigma_k(x) = \hat{\sigma}(x, kx)$, for some $\hat{\sigma} > 0$ on T^2.

When $k<1$ is irrational, it has a unique continued fraction expansion[15]

$$k=[a_1, a_2, \ldots]=\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}} \tag{5.2}$$

with positive integers $a_1, a_2, \ldots$. Truncations of this expansion provide the "best" rational approximants k_n to k,

$$k_n=\frac{p_n}{q_n}=[a_1, \ldots, a_n] \tag{5.3}$$

They are best in the sense that if for some n, $|k-p/q|<|k-p_n/q_n|$, then $q>q_n$. It is when k satisfies

$$|k-p_n/q_n|<1/q_n^\gamma \qquad \forall n \tag{5.4}$$

for large enough $\gamma>0$ that $\sigma^*(k, L)$ can be shown to have "plateau structure." The larger γ is, the faster the a_n grow to infinity, and the longer the plateaus are, as we now explain.

For any particular rational approximant $k_n=p_n/q_n$ we have from (5.1)

$$[\sigma^*(k_n, L)]^{-1}=[\sigma^*(k_n)]^{-1}+L_n\cdot O(1/L) \tag{5.5}$$

where $L_n=q_n$ is the period of $\sigma_{k_n}(x)$. Furthermore, for smooth $\hat{\sigma}$ on T^2 there is a $C>0$ such that

$$|1/\sigma^*(k, L)-1/\sigma^*(k_n, L)| \leqslant CL\,|k-k_n| \tag{5.6}$$

Now let $\bar{\sigma}$ be the value of σ^* for irrational k and let

$$\varepsilon_n=|\bar{\sigma}-\sigma^*(k_n)|>0 \tag{5.7}$$

Choose A_n so large that for $L>A_n$,

$$|\sigma^*(k_n, L)-\sigma^*(k_n)| << \varepsilon_n \tag{5.8}$$

which by (5.5) will be satisfied when

$$A_n >> L_n/\varepsilon_n \tag{5.9}$$

Next pick $B_n >> A_n$. When k is so close to k_n that

$$CB_n\,|k-k_n| << \varepsilon_n \tag{5.10}$$

as well, then by (5.5) and (5.6) $|\sigma^*(k, L)-\sigma^*(k_n)| << |\bar{\sigma}-\sigma^*(k_n)|$ for $A_n < L < B_n$, so that the graph of $\sigma^*(k, L)$ has a "plateau" for L in this range. This closeness of k to k_n can be arranged by requiring that a_{n+1} be sufficiently large, or equivalently, by demanding that γ be large. Clearly, the smaller $|k-k_n|$ is, the longer the plateau.

As a specific example, we consider the checkerboard of Theorem 3.1, where we know the $\sigma_n^* = \sigma^*(k_n)$ exactly,

$$\frac{1}{\sigma_n^*}=\frac{1}{2}-\frac{1}{k_n q_n^2} \tag{5.11}$$

for $k_n = p_n/q_n$ with p_n and q_n odd and relatively prime. Moreover, though $\hat{\sigma}$ is not smooth, (5.6) can nonetheless be shown to hold. In order to guarantee that $1/\sigma^*(k, L)$ is within, say, $1/q_n^4$ to $1/\sigma_n^*$, our choices for A_n and B_n must satisfy

$$\frac{C_1 L}{q_n^\gamma}+\frac{C_2 q_n}{L}<\frac{1}{q_n^4}, \qquad A_n < L < B_n \tag{5.12}$$

for some C_1, $C_2 > 0$. We can then choose, for example, $A_n = q_n^6$ and $B_n = q_n^{\gamma-5}$, obtaining plateaus when $\gamma > 11$ in (5.4).

APPENDIX. EXPLICIT CALCULATION OF THE EFFECTIVE CONDUCTIVITY FOR A ONE-DIMENSIONAL MODEL [PROOF OF THEOREM 3.1]

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We first consider k irrational. By ergodicity, $1/\sigma^*(k)$ is given by (2.9), which clearly has value 1/2.

Now let $k = p/q$, where p and q are relatively prime integers. The orbit having slope k and beginning at the origin in T^2 can be represented as the diagonal line of the rectangle $R(q, p) = [0, q] \times [0, p]$. The rectangle $R(q, p)$ is equipped with a checkerboard grid consisting of $4pq$ squares having sides of length 1/2, where the square closest to the origin has $\sigma = \sigma_2$; see Fig. 1. We must compute $p_1(k)$: the proportion of the length of the diagonal spent in regions having $\sigma = \sigma_1$. Observe that the square closest to (q, p) has $\sigma = \sigma_2$. Thus, by symmmetry, it suffices to consider that half of the diagonal lying in $R(q/2, p/2)$. In order to work with integers rather than half-integers, it is convenient to now double the length scales by mapping $R(q/2, p/2)$ to $R'(q, p)$, equipped with a checkerboard which has pq *unit* squares.

Note that if either p or q is even (but one of them is odd, since p and q are relatively prime), then the square closest to the vertex (q, p) has $\sigma = \sigma_1$. Simple symmetry considerations now show that $p_1(k) = p_2(k) = 1/2$.

We now take up the case where p and q are two relatively prime odd numbers. Without loss of generality it may be assumed that $q > p$, since $\sigma^*(p/q) = \sigma^*(q/p)$. The regions $\sigma = \sigma_1$ and $\sigma = \sigma_2$ along the diagonal from $(0, 0)$ to (q, p) each consist of several intervals. Hence, to determine $p_1(k)$, we merely have to find the endpoints of all of these intervals and then decide which intervals have $\sigma = \sigma_1$.

Observe that a change in the conductivity along the diagonal can only occur when one of the integer lines $x = i$ $(i = 1,..., q-1)$ or $y = j$ $(j = 1,..., p-1)$ is crossed. Furthermore, since the only points on the diagonal having integer coordinates are $(0, 0)$ and (q, p), it follows that the conductivity must change whenever an integer line is crossed. The vertical integer lines can be used to divide the diagonal q segments: the ith segment has x coordinates between $i-1$ and i for $i = 1,..., q$. Each segment either has a single conductivity (if the segment crosses no horizontal integer line) or it has a single change of conductivity (if the segment crosses a horizontal integer line). No segment can have two or more changes of conductivity.

We first treat the $p-1$ segments for which there is a change in the conductivity. Our basic tool is the following fact: if both p and q are odd and if i, k, and l are chosen so that $lq = kp + i$, then $k + l$ has the same parity as i.

Now for each $i = 1, 2,..., p-1$, let k_i and l_i be the smallest nonnegative integers for which $l_i q = k_i p + i$. The numbers k_i and l_i have a geometric interpretation: the segment of the diagonal having x projection $[k_i, k_i + 1]$ intersects the horizontal integer line $y = l_i$, and the intersection occurs at $(k_i + 1/p, l_i)$. Note that the diagonal crosses integer lines for the $(k_i + i/p, l_i)$, respectively. Because $k_i + l_i$ and i have the same parity, and because the first conductivity seen along the diagonal is $\sigma = \sigma_2$, it follows that if i is odd, then $\sigma = \sigma_2$ for $k_i < x < k_i \pm i/p$ and $\sigma = \sigma_1$ for $k_i + i/p < x < k_i + 1$. By the same reasoning, the conductivities are reversed when i is even. Letting s_i denote the fraction of the segment having x projection $[k_i, k_i + 1]$ for which $\sigma = \sigma_1$, we see that $s_i = (p - i)/p$ for i odd and $s_i = i/p$ for i even.

We now return to investigate the $q - p + 1$ segments for which there was no change in the conductivity. We claim that $\sigma = \sigma_2$ for $\frac{1}{2}(q - p) + 1$ of these, and that $\sigma = \sigma_1$ for the remaining $\frac{1}{2}(q - p)$. To verify the claim, one observes that if the segment having x projection $[k, k + 1]$ has only one conductivity, then that segment lies between two successive horizontal integer lines, say $y=l$ and $y=l+1$. Let $j=kp-lq$; then the diagonal crosses the vertical integer lines $x=k$ and $x=k+1$ at $(k, l+j/q)$ and $(k+1, l+(p+j)/q)$. The condition that the segment not cross a horizontal integer line for x between k and $k+1$ implies that $j=0,..., q-p$. Note that these values of j account for all $q-p+1$ segments having only one conductivity. Furthermore, because j and $k+l$ have the same parity, and because the conductivity changes for the $(k+l)$th time at the beginning of the segment, it follows that $\sigma=\sigma_2$ if j is odd.

A direct calculation now shows that if p and q are relatively prime and odd, then

$$\begin{aligned}
p_1\left(\frac{p}{q}\right) &= \frac{1}{q}\left(\frac{q-p}{2}+\sum_{i=1}^{p-1} s_i\right) \\
&= \frac{1}{2}-\frac{1}{2q}\left\{p-2\sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2}\left(1-\frac{i}{p}\right)-2\sum_{\substack{i=2 \\ i \text{ even}}}^{p-1}\frac{i}{p}\right\} \\
&= \frac{1}{2}-\frac{1}{2q}\left\{p-2\left[\frac{p-1}{2}-\frac{1}{p}\left(\frac{p-1}{2}\right)^2\right]+\frac{p^2-1}{4p}\right\} \\
&= \frac{1}{2}-\frac{1}{2pq}
\end{aligned}$$

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